

# A Variant of Newton – Raphson Method upto Fourth Order Convergence

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**Abstract-** In this paper, we suggest a variant of Newton – Raphson method with which one can attain up to fourth order convergence provided the difference between  $f'(x_0)$  and  $f'(\alpha)$  is negligible up to the desired accuracy, where ' $\alpha$ ' is the real root of  $f(x)=0$ .

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**Keywords:** Newton's Formula, Nonlinear Equation, Iterative Methods, Order of Convergence, Function Evaluations.

## I. INTRODUCTION

The most widely used algorithm for solving a nonlinear equation in one variable i.e;

$$f(x)=0 \quad (1.1)$$

Where  $f : D \subset R \rightarrow R$  is a scalar function on an open interval D, by the use of value of function and its derivative is the well known Newton's method given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (n=0, 1, 2, \dots) \quad (1.2)$$

which has a quadratic convergence.

As is known that derivation of Newton's method involves an indefinite integral of the derivative of the function, and the relevant area is approximated by a rectangle. But, by approximating this indefinite integral by a trapezoid instead of a rectangle, Weerakoon and Fernando [4] suggested a Variant of Newton's method which requires three functional evaluations at each step, given by

$$x_{n+1} = x_n - \frac{2f(x_n)}{[f'(x_n) + f'(x_{n+1}^*)]} \quad (n=0, 1, 2, \dots) \quad (1.3)$$

$$\text{where } x_{n+1}^* = x_n - \frac{f(x_n)}{f'(x_n)} \quad (1.4)$$

which has a cubic rate of convergence.

In this paper, we suggest a Variant of Newton-Raphson method which requires only two functional evaluations at each iteration apart from one extra evaluation initially. In section 2, we suggest the Variant of Newton-Raphson method whereas the rate of convergence of this method is obtained in section 3. In the concluding section, we give some numerical examples to exhibit the superiority of this Newton-Raphson's variant.

## II. VARIANT OF NEWTON – RAPHSON METHOD

Let  $x_0$  be an initial approximation to the real root ' $\alpha$ ' of the equation (1.1) is an open interval in which  $f(x)$  is continuous and let

$$x_0^* = x_0 + h \quad (2.1)$$

and  $x_1$  be the next two successive approximations.

If  $x_0$  be in the vicinity of ' $\alpha$ ' and if ' $h$ ' is small enough, then as derived in the Newton's method, one can have

$$x_0^* = x_0 - \frac{f(x_0)}{f'(x_0)} \quad (2.2)$$

and then the next approximate ' $x_1$ ' is to be obtained as

$$x_1 = x_0^* - \frac{f(x_0^*)}{f'(x_0)} \quad (2.3)$$

by fixing  $f'(x_0)$  and  $x_0^*$  is as given in (2.2).

Again, keeping  $f'(x_0)$  unchanged, the next approximation  $x_2$  to be obtained as

$$x_2 = x_1^* - \frac{f(x_1^*)}{f'(x_0)} \quad (n=0, 1, 2, \dots), \quad (2.4)$$

where  $x_1^* = x_1 - \frac{f(x_1)}{f'(x_0)}$ .

In general, this variant of Newton – Raphson method can be defined as

$$x_{n+1} = x_n^* - \frac{f(x_n^*)}{f'(x_0)} \quad (n=0, 1, 2, \dots) \quad (2.5)$$

where  $x_n^* = x_n - \frac{f(x_n)}{f'(x_0)}$ , which requires only two functional evaluations at each step leaving the one  $f'(x_0)$  calculated initially.

**Algorithm 2.1:** For a given  $x_0$ , compute the approximate solution  $x_{n+1}$  by the iterative scheme:

$$x_{n+1} = x_n^* - \frac{f(x_n^*)}{f'(x_0)} \quad \text{where} \quad x_n^* = x_n - \frac{f(x_n)}{f'(x_0)}$$

This algorithm requires only two functional evaluations at each step leaving the one derivative  $f'(x_0)$  calculated initially. Also, the efficiency index of the method  $\sqrt[2]{4}$  i.e the order of the method is four provided the difference between  $f'(\alpha)$  and  $f'(x_0)$  where ' $\alpha$ ' the exact root of (1.1) be, is zero up to the desired accuracy neglecting evaluation of  $f'(x_0)$  only once initially.

## III. RATE OF CONVERGENCE OF VARIANT OF NEWTON – RAPHSON METHOD

**Theorem 3.1:** Let  $\alpha \in D$  be a simple zero of the function  $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$  in an open interval  $D$ . If  $x_0$  is in the very vicinity of the root ' $\alpha$ ' then the algorithm (2.1) has fourth order convergence provided  $\frac{f'(\alpha)}{f'(x_0)}$  is nearly unity up to the desired accuracy.

**Proof:** If ' $\alpha$ ' be the exact solution of the eqn. (1.1), then

$$f(\alpha) = 0 \quad (3.1)$$

Let  $e_{n+1}$  and  $e_n$  be the errors at  $(n+1)^{\text{th}}$  and  $n^{\text{th}}$  stages and let  $x_{n+1}$  and  $x_n$  be the  $(n+1)^{\text{th}}$  and  $n^{\text{th}}$  approximations to the root ' $\alpha$ ' of the eqn. (1.1). Therefore, we have

$$X_{n+1} = e_{n+1} + \alpha \quad (3.2)$$

$$X_n = e_n + \alpha \quad (3.3)$$

Now,  $f(X_n) = f(\alpha + e_n)$

$$\begin{aligned} &= f'(\alpha) \left[ e_n + \frac{1}{2!} \frac{f''(\alpha)}{f'(\alpha)} e_n^2 + \frac{1}{3!} \frac{f'''(\alpha)}{f'(\alpha)} e_n^3 + \frac{1}{4!} \frac{f^{iv}(\alpha)}{f'(\alpha)} e_n^4 + O(e_n^5) \right] \\ &= f(\alpha) + f'(\alpha) e_n + \frac{f''(\alpha)}{2!} e_n^2 + \frac{f'''(\alpha)}{3!} e_n^3 + \frac{f^{iv}(\alpha)}{4!} e_n^4 + O(e_n^5) \\ &= f'(\alpha) [e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + O(e_n^5)] \end{aligned} \quad (3.4)$$

Where  $c_j = \frac{1}{j!} \frac{f^{(j)}(\alpha)}{f'(\alpha)}, \quad (j=2, 3, 4, \dots)$

Now again,  $X_n^* = X_n - \frac{f(X_n)}{f'(X_0)}$

$$\begin{aligned} &= \alpha + e_n - \frac{f'(\alpha)}{f'(X_0)} [e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + O(e_n^5)] \quad (\text{by 3.3 and 3.4}) \\ &= \alpha + (1-k)e_n - c_2 k e_n^2 - c_3 k e_n^3 - c_4 k e_n^4 + O(e_n^5), \end{aligned} \quad (3.5)$$

where  $k = \frac{f'(\alpha)}{f'(X_0)} \quad (3.6)$

$$\begin{aligned} \therefore f(X_n^*) &= f(\alpha) + f'(\alpha) [(1-k)e_n - c_2 k e_n^2 - c_3 k e_n^3 - c_4 k e_n^4 + O(e_n^5)] \\ &+ \frac{f''(\alpha)}{2!} [(1-k)^2 e_n^2 + c_2^2 k^2 e_n^4 + 2\{-k(1-k)c_2 e_n^3 - k(1-k)c_3 e_n^4\} + O(e_n^5)] \\ &+ \frac{f'''(\alpha)}{3!} [(1-k)^3 e_n^3 + 3\{-k(1-k)^2 c_2 e_n^4 - k^2(1-k)c_2^2 e_n^4 + O(e_n^5)\}] \\ &+ \frac{f^{iv}(\alpha)}{4!} [(1-k)^4 e_n^4 + O(e_n^5)] + \dots \end{aligned}$$

$$\begin{aligned} &= f'(\alpha)(1-k)e_n - [f'(\alpha)c_2 k - \frac{f''(\alpha)}{2}(1-k)^2] e_n^2 \\ &- [f'(\alpha)c_3 k + f''(\alpha)k(1-k)c_2 - f'''(\alpha)(1-k)^3] e_n^3 \\ &- [f'(\alpha)c_4 k + f''(\alpha)k(1-k)c_3 + \frac{f'''(\alpha)}{2}k(1-k)^2 c_2 + \frac{f''(\alpha)}{2}c_2^2 k^2 \\ &- \frac{f^{iv}(\alpha)}{4!}(1-k)^4] e_n^4 + O(e_n^5) \\ &= f'(\alpha)(1-k)e_n - [f'(\alpha)c_2 k - \frac{f''(\alpha)}{2}(1-k)^2] e_n^2 \\ &- [f'(\alpha)c_3 k + f''(\alpha)k(1-k)c_2 - f'''(\alpha)(1-k)^3] e_n^3 \\ &- [f'(\alpha)c_4 k + f''(\alpha)k((1-k)c_3 + \frac{c_2^2 k}{2}) + \frac{f'''(\alpha)}{2}k(1-k)^2 c_2 \\ &- \frac{f^{iv}(\alpha)}{4!}(1-k)^4] e_n^4 + O(e_n^5) \end{aligned} \quad (3.7)$$

Now,

$$\begin{aligned}
 x_n^* - \frac{f(x_n^*)}{f'(x_0)} &= \alpha + (1-k)e_n - c_2 k e_n^2 - c_3 k e_n^3 - c_4 k e_n^4 \\
 &\quad - \frac{f''(\alpha)}{f'(x_0)}(1-k)e_n + \left[ \frac{f''(\alpha)}{f'(x_0)} c_2 k - \frac{f''(\alpha)}{2f'(x_0)}(1-k)^2 \right] e_n^2 \\
 &\quad + \left[ \frac{f''(\alpha)}{f'(x_0)} c_3 k + \frac{f''(\alpha)}{f'(x_0)} k(1-k)c_2 - \frac{f'''(\alpha)}{f'(x_0)}(1-k)^3 \right] e_n^3 \\
 &\quad + \left[ \frac{f''(\alpha)}{f'(x_0)} c_4 k + \frac{f''(\alpha)}{f'(x_0)} k((1-k)c_3 + \frac{c_2^2 k}{2}) + \frac{f'''(\alpha)}{2f'(x_0)} k(1-k)^2 c_2 \right. \\
 &\quad \left. - \frac{f^{iv}(\alpha)}{4!f'(x_0)}(1-k)^4 \right] e_n^4 + O(e_n^5)
 \end{aligned}$$

By (2.5), we have

$$\begin{aligned}
 x_{n+1} &= \alpha + (1-k)e_n - c_2 k e_n^2 - c_3 k e_n^3 - c_4 k e_n^4 \\
 &\quad - k(1-k)e_n + \left[ c_2 k^2 - \frac{f''(\alpha)}{2f'(x_0)}(1-k)^2 \right] e_n^2 \\
 &\quad + \left[ c_3 k^2 + \frac{f''(\alpha)}{f'(x_0)} k(1-k)c_2 - \frac{f'''(\alpha)}{f'(x_0)}(1-k)^3 \right] e_n^3 \\
 &\quad + \left[ c_4 k^2 + \frac{f''(\alpha)}{f'(x_0)} k((1-k)c_3 + \frac{c_2^2 k}{2}) + \frac{f'''(\alpha)}{2f'(x_0)} k(1-k)^2 c_2 \right. \\
 &\quad \left. - \frac{f^{iv}(\alpha)}{4!f'(x_0)}(1-k)^4 \right] e_n^4 + O(e_n^5)
 \end{aligned}$$

Where ' $k$ ' is as given in (3.6).

$$\begin{aligned}
 &= \alpha + (1-k)^2 e_n - [k(1-k)c_2 + \frac{f''(\alpha)}{2f'(x_0)}(1-k)^2] e_n^2 \\
 &\quad - [k(1-k)c_3 - \frac{f''(\alpha)}{f'(x_0)} k(1-k)c_2 + \frac{f'''(\alpha)}{f'(x_0)}(1-k)^3] e_n^3 \\
 &\quad - [k(1-k)c_4 - \frac{f''(\alpha)}{f'(x_0)} k((1-k)c_3 + \frac{c_2^2 k}{2}) - \frac{f'''(\alpha)}{2f'(x_0)} k(1-k)^2 c_2 \\
 &\quad + \frac{f^{iv}(\alpha)}{4!f'(x_0)}(1-k)^4] e_n^4 + O(e_n^5)
 \end{aligned} \tag{3.8}$$

Therefore, by (3.8), (3.2) the method (2.5) takes the form

$$\begin{aligned}
 \alpha + e_{n+1} &= \alpha + (1-k)^2 e_n - [k(1-k)c_2 + \frac{f''(\alpha)}{2f'(x_0)}(1-k)^2] e_n^2 \\
 &\quad - [k(1-k)c_3 - \frac{f''(\alpha)}{f'(x_0)} k(1-k)c_2 + \frac{f'''(\alpha)}{f'(x_0)}(1-k)^3] e_n^3 \\
 &\quad - [k(1-k)c_4 - \frac{f''(\alpha)}{f'(x_0)} k((1-k)c_3 + \frac{c_2^2 k}{2}) - \frac{f'''(\alpha)}{2f'(x_0)} k(1-k)^2 c_2 \\
 &\quad + \frac{f^{iv}(\alpha)}{4!f'(x_0)}(1-k)^4] e_n^4 + O(e_n^5)
 \end{aligned} \tag{3.9}$$

$$\begin{aligned}
 \therefore e_{n+1} &= (1-k)^2 e_n - [(1-k)c_2 k + \frac{f''(\alpha)}{2f'(x_0)}(1-k)^2] e_n^2 \\
 &\quad - [(1-k)c_3 k - \frac{f''(\alpha)}{f'(x_0)} k(1-k)c_2 + \frac{f'''(\alpha)}{f'(x_0)}(1-k)^3] e_n^3 \\
 &\quad - [(1-k)c_4 k - \frac{f''(\alpha)}{f'(x_0)} k((1-k)c_3 + \frac{c_2^2 k}{2}) - \frac{f'''(\alpha)}{2f'(x_0)} k(1-k)^2 c_2 \\
 &\quad + \frac{f^{iv}(\alpha)}{4!f'(x_0)}(1-k)^4] e_n^4 + O(e_n^5)
 \end{aligned} \tag{3.10}$$

It can be seen from the above step that the terms up to and including  $e_n^3$  vanish if 'k' is nearly unity up to the desired accuracy and if it does so, we have

$$e_{n+1} = -\frac{c_2^2 k^2}{2} \frac{f''(\alpha)}{f'(x_0)} e_n^4 + O(e_n^5) \quad (3.11)$$

That means  $e_{n+1} \propto e_n^4$  (3.12)

$\therefore$  The method (2.5) has a fourth order convergence if  $k=1$ .

#### IV. NUMERICAL EXAMPLES

We consider few numerical examples considered by Weerakoon and Fernando [4] and by Grewal [1] and the method (2.5) is compared with the methods (1.2) and (1.3). The computational results are tabulated below and the results are correct up to an error less than  $5 \times 10^{-7}$ .

Table 1

Function	$x_0$	$i$			NOFE			Root
$f(x)$		NM	VNM	VNR	NM	VNM	VNR	
(1) $e^{x-1} + x - 3$	1.4	3	2	2	6	6	5	1.442854
	1.5	3	2	2	6	6	5	
(2) $2x - \log_{10} x - 7$	3	3	2	2	6	6	5	1.404492
	3.1	3	2	2	6	6	5	
(3) $x \log_{10} x - 1.2$	2.5	3	2	2	6	6	5	0.25753028
	2.6	3	2	2	6	6	5	
(4) $\cos(x) - x$	0.7	2	2	2	4	6	5	0.7390851
	0.8	3	2	2	6	6	5	
(5) $x^2 - e^x - 3x + 2$	0.3	2	2	2	4	6	5	0.2575303
	0.4	3	2	2	6	6	5	

NM – Newton's Method

VNM – Variant of Newton's Method

VNR – Variant of Newton – Raphson

NOFE – Number of Function Evaluations

i-Number of iterations to approximate the root to 7 decimal places

#### V. CONCLUSION

From the above tabulated values, we conclude that the method (2.5) has an advantage of more rapid convergence with only two functional evaluations from second iteration onwards, over the other methods discussed in this paper.

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